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isoelastic demand and constant unit costs"**

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# New properties of the Cournot duopoly with isoelastic demand and constant unit costs

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Abstract.

The object of the work is to perform the global analysis of the Cournot duopoly model with isoelastic demand function and unit costs, presented in Puu (1991). The bifurcation of the unique Cournot fixed point is established, which is a resonant case of the Neimark-Shacker bifurcation. New properties associated with the introduction of horizontal branches are evidenced. These properties differ significantly when the constant value is zero or positive and small. The good behavior of the case with positive constant is proved, leading always to positive trajectories. Also when the Cournot fixed point is unstable, stable cycles of any period may exist.

**Keywords:** Cournot duopoly; isoelastic demand function; multistability; border-collision bifurcations.

**JEL classification codes:** C15; C62; D24; D43

## 1 Introduction

Rand (1978) suggested that Cournot duopolies, if they were characterized by reaction functions of upside-down U-shape, might provide for multiple coexistent Cournot equilibria, and, depending on parameters, display many of the phenomena known from complex dynamics in other fields. Rand, however, supplied no substantial assumptions, based on economic theory, from which such reaction functions could arise.

Everyone who worked with this knows that it is not easy, because, except having the assumptions based on generally accepted microeconomics, one would like to be able to solve for the reaction functions in explicit closed form, be able to calculate the coordinates of Cournot

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equilibria, and yet find some phenomena economically interesting to investigate. Economic theory abounds of useful assumptions that might lead to interesting reaction function shapes, but very few allow one to fulfil the programme of actual closed form derivations.

One of the present authors suggested in 1991 (Puu, 1991) the combination of isoelastic demand, market price and quantity being related by reciprocity, with constant marginal costs for the competitors; this allowed one to make the explicit derivations, and resulted in, if not multiple Cournot equilibria, at least the general shapes that Rand wanted, producing period doubling bifurcation cascades to chaos. The model has since then been used in a sizeable number of publications (see the books by Puu and Sushko, 2002 and Bischi et al., 2009) and several models were generalized by using adaptive rules or heterogeneous participants (Kopel, 1996, Puu, 1998, Ahmed et al., 2000, Agiza and Elsadany, 2004, Puu and Sushko, 2006, Agliari, 2006, Agliari et al., 2006a,b, Angelini et al., 2009, Tramontana, 2010, to cite a few, and also using chaos control, as in Matsumoto, 2006, Chen and Chen, 2007).

The isoelastic demand function has its advantages and disadvantages. The advantages are that it results when the consumers optimize general utility functions of Cobb-Douglas shape. Consumers then always spend constant budget shares on each commodity, which provides for the reciprocity of price and quantity. As further all consumers have demand functions of the same shape, this provides for one of the few cases where the aggregation problem is easily solved and a market demand function of the same shape results.

The disadvantages are that the model is no good for dealing with monopoly. As price and quantity are reciprocal, the revenue of a monopolistic firm would be constant, no matter how much the firm sells. On the other hand, any reasonable production cost function increases with output; so producing nothing is the best choice for lowering costs. With constant revenue, the obvious best choice is to actually produce nothing, so avoiding costs, and selling this nothing at an infinite price. The solution has no meaning in terms of substance; it is purely formal. Ultimately it results from the unlimited substitution possibilities inherent in the Cobb-Douglas indifference curves, and so illustrates the difficulty of finding assumptions that in a reasonable way represent the phenomena globally. The same problem arises in the case of collusion.

In duopoly the problem does not arise in a direct manner, but it is there, and this paper is in a way dealing with precisely it. As the unimodal reaction functions eventually come down to the axes, and as negative supplies make no sense, a first choice is to replace negative values with a zero branch. Negative supplies would also be related to

negative profits, and so it is natural to assume that after the reaction function comes down to the axis the firm produces nothing. However, once one axis is hit, the system can end up at the origin where the reaction functions also intersect, i.e., at the collusion state. This is, however, forbidden by law in most countries. Further, the reaction functions intersect with infinite slope in the origin, so it is totally unstable, and the system would be thrown away by any slight disturbance.

Yet, solutions involving the zero branches are there and even become stable in a weak Milnor sense. This has never been properly investigated, and the first part of the present paper deals with this.

One can also avoid the origin through stipulating that the duopolists do not actually close down when they cannot make any profit, but keep to some small "epsilon" stand-by output. This assumption was originally introduced in Puu (1991) to the end of keeping the computer from sticking to a totally unstable origin in numerical work, but it makes sense also in terms of substance. The importance of the numerical value of this "epsilon" stand-by output, has never been investigated, and is the purpose of the present study in the second part of this paper.

It is worth noting that the resulting reaction functions are piecewise smooth, and we can apply the arguments of the model here considered, also to several other models, proposed for example in (Puu and Sushko, 2002 and Bischi et al., 2009) as well as in many other duopoly or oligopoly models.

The plan of the work is as follows. In Section 2 we shall recall the model, considering the case in which the reaction functions are defined with a zero branch, and the global dynamics associated with these branches is studied. Clearly they play a role after the final bifurcation, when all the trajectories are mapped into the invariant coordinate axes, but also before, when the Cournot equilibrium is locally stable. Moreover, the true nature of the bifurcation of the unique Cournot fixed point is established, which is a resonant case of the Neimark-Shacker bifurcation. Then in Section 3 we shall consider the modified model in which the zero branch of the reaction functions is changed into a small positive constant value. This economically plausible change leads to dynamics which are always positive. The states previously convergent to the axes now are convergent to some cycle in the positive phase space. As we shall see, also after the final bifurcation the dynamics are convergent to a unique superstable cycle, whose period may be any integer number, depending on the parameters and on the small constant value assumed in the model. Section 4 concludes, noticing that the global analysis here performed also works for a continuous piecewise-linear model, with horizontal graphs in the reaction functions, as well as in other duopoly

models in which the constraint of an horizontal branch is assumed.

## 2 The basic Cournot model with isoelastic demand function

Assume, as in Puu (1991), the inverse demand function

$$p = \frac{1}{x + y}, \quad (1)$$

where  $p$  denotes market price and  $x, y$  denote the outputs of the duopolists. Given the competitors have constant marginal costs, denoted  $a, b$  respectively, the profits are

$$U = \frac{x}{x + y} - ax, \quad (2)$$

$$V = \frac{y}{x + y} - by. \quad (3)$$

Putting the derivatives  $\partial U / \partial x = 0$  and  $\partial V / \partial y = 0$ , and solving for  $x, y$ , one obtains

$$x' = \sqrt{\frac{y}{a}} - y, \quad (4)$$

$$y' = \sqrt{\frac{x}{b}} - x, \quad (5)$$

as the reaction functions. The dash, as usual, represents the next iterate, i.e., the "best reply" of one competitor given the observed supply of the other.

Obviously, (4) returns a negative reply  $x'$  if  $y > 1/a$ , and (5) a negative reply  $y'$  if  $x > 1/b$ . To avoid this, we put  $x' = 0$  whenever  $y > 1/a$ , and  $y' = 0$  whenever  $x > 1/b$ . This means reformulating (4)-(5) as a continuous piecewise smooth map  $T$ ,  $T(x, y) = (x', y')$  defined as follows:

$$x' = f(y) = \begin{cases} \sqrt{\frac{y}{a}} - y & \text{if } 0 \leq y \leq \frac{1}{a} \\ 0 & \text{if } y > \frac{1}{a} \end{cases}, \quad (6)$$

$$y' = g(x) = \begin{cases} \sqrt{\frac{x}{b}} - x & \text{if } x \leq \frac{1}{b} \\ 0 & \text{if } x > \frac{1}{b} \end{cases}. \quad (7)$$

As we know, the intersections of the two reaction functions lead to the Nash equilibria, and in this basic model by Cournot equilibrium we indicate the unique one with positive coordinates, given by

$$C = (x_C^*, y_C^*) = \left( \frac{b}{(a + b)^2}, \frac{a}{(a + b)^2} \right) \quad (8)$$

There is also the origin  $O = (0, 0)$  which is a locally unstable equilibrium. As shown in Bischi et al. (2000), the dynamic behaviors of a duopoly model can be studied via the one-dimensional map  $x' = F(x) = f(g(x))$  that in our case is piecewise smooth. An example is shown in Fig. 1a at parameters' values for which the Cournot fixed point is locally stable.

Considering the case there shown, from the existence of two fixed points of  $F(x)$ , the locally unstable origin  $O = (0, 0)$  and the Cournot point  $C = (x_C^*, y_C^*)$  (here locally stable) we know that also a 2-cycle exists on the coordinate axes, say  $C_{2A}$ , given by  $\{(x_C^*, 0), (0, y_C^*)\}$  which is locally a saddle.

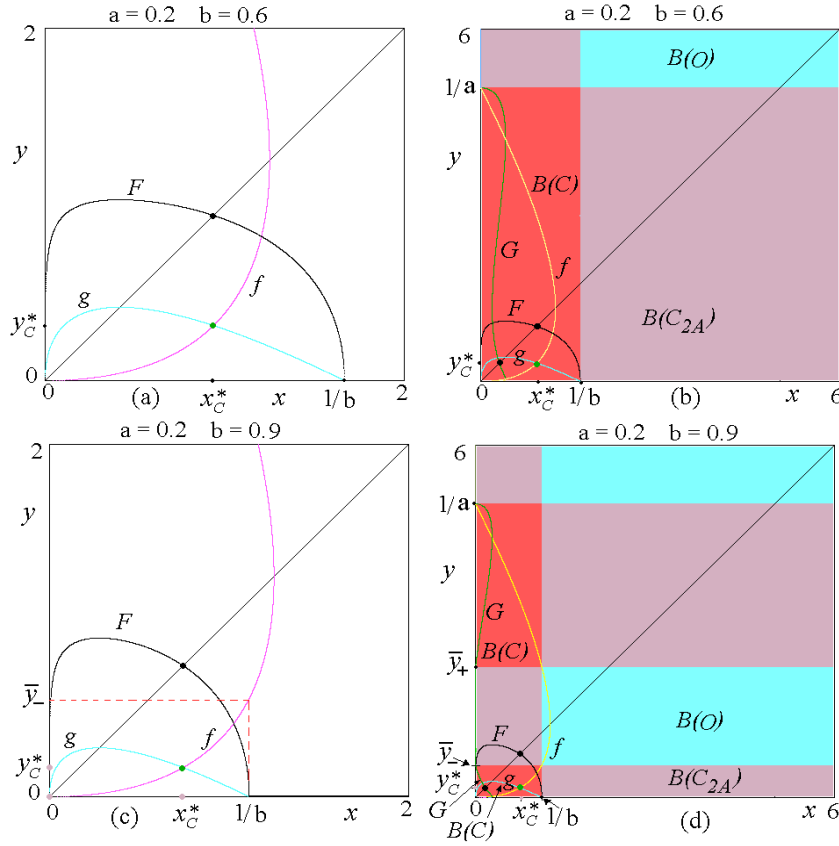


Fig. 1

From the main property of the Cournot models (to have a separate second iterate function) the basin of attraction of the Cournot fixed point for the two-dimensional map  $T(x, y)$  is given by the Cartesian product  $\mathcal{B}_T(x_C^*, y_C^*) = \mathcal{B}_F(x_C^*) \times \mathcal{B}_G(y_C^*)$  where  $\mathcal{B}_F(x_C^*)$  is the basin of attraction of the stable fixed point  $x_C^*$  for the map  $F(x)$  and  $\mathcal{B}_G(y_C^*)$  is the basin of attraction of the stable fixed point  $y_C^*$  for the map  $G(y) = g \circ f(y)$ . In the case shown in Fig. 1a we have that  $\mathcal{B}_F(x_C^*)$  is the whole segment  $]0, 1/b[$  and  $\mathcal{B}_G(y_C^*)$  is the whole segment  $]0, 1/a[$ . It follows that the basin of

attraction of the Cournot fixed point for the two-dimensional map  $T(x, y)$  is given by the Cartesian product  $\mathcal{B}_T(x_C^*, y_C^*) = \mathcal{B}_F(x_C^*) \times \mathcal{B}_G(y_C^*) = ]0, 1/b[ \times ]0, 1/a[$  as shown in Fig. 1b. We can see that all the other points of the phase plane are mapped on the coordinate axes, either in the fixed point  $O$  or converging to the 2-cycle saddle  $C_{2A}$ .

This is not in contradiction with the fact that these cycles are locally unstable. From a dynamical point of view these cycles (the origin  $O$  and the saddle  $C_{2A}$ ) are called stable *in weak sense* or *in Milnor sense* (see Milnor, 1985<sup>2</sup>). The reason why these cycles are stable in Milnor sense is the existence of "zero-branches" in the definition of the maps  $F(x)$  and  $G(y)$ . For the one-dimensional map  $F(x)$  all the points in  $]1/b, +\infty[$  are mapped into the origin, and thus also in the one-dimensional case (map  $F(x)$ ) the basin of the origin is of positive measure. Similarly for  $G(y)$ . The separators between the basin of the proper attracting set (now  $C$ ) and those in Milnor sense is given by the lines  $x = \frac{1}{b}$  and  $y = \frac{1}{a}$  as long as the nonlinear graphs of the two reaction functions are included in the rectangle  $Q = [0, 1/b] \times [0, 1/a]$ .

The structure of the basins have a first change (global bifurcation) as soon as one of the two reaction functions exits from  $Q$ , thus modifying the structure of the composite maps  $F(x)$  or  $G(y)$ . Let us increase the parameter  $b$  keeping fixed the value of the parameter  $a$  (with obvious changes can be dealt with the symmetric case in which we increase  $a$ ). Noticing that the maximum of the function  $f(y)$  occurs at  $y_{cr} = \frac{1}{4a}$  (as  $f'(y_{cr}) = 0$ ) and  $f(y_{cr}) = y_{cr}$ , we have that it is also a critical point of  $G(y)$  (here a local minimum) as  $G'(y_{cr}) = 0$  and  $G(y_{cr}) = \sqrt{\frac{1}{4ab}} - \frac{1}{4a}$ . Then the minimum of  $G(y)$  reaches zero at the same time in which the function  $f(y)$  reaches the value  $\frac{1}{b}$ . In fact  $f(y_{cr}) = \frac{1}{4a} = \frac{1}{b}$  and  $G(y_{cr}) = \sqrt{\frac{1}{4ab}} - \frac{1}{4a} = 0$  both occur when the parameters satisfy

$$\sqrt{\frac{b}{a}} = 2 \quad , \quad r = \frac{b}{a} = 4 \quad (9)$$

in our example (with  $a = 0.2$ ) this bifurcation in the basins occurs at  $b = 0.8$ . In fact, after this contact the structure of the basins changes. An example is shown in Fig. 1c,d at  $b = 0.9$ . The function  $F(x)$  is qualitatively the same while  $G(y)$  now has a zero branch (as shown in Fig. 1d). We notice that the Cournot fixed point is still stable, and its basin of attraction is always given by  $\mathcal{B}_T(x_C^*, y_C^*) = \mathcal{B}_F(x_C^*) \times \mathcal{B}_G(y_C^*)$ , where  $\mathcal{B}_F(x_C^*)$  is always the whole segment  $]0, 1/b[$  while  $\mathcal{B}_G(y_C^*)$  now consists

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<sup>2</sup>a cycle is said stable in Milnor sense if it is locally unstable but its basin of attraction is of positive measure in the phase space.

of two disjoint intervals. Now the basin  $\mathcal{B}_G(y_C^*)$  can be obtained by using the inverse of the function  $f(y)$  as follows:  $\mathcal{B}_G(y_C^*) = f^{-1}(\mathcal{B}_F(x_C^*))$  and in our case the basin  $\mathcal{B}_G(y_C^*) = f^{-1}(]0, 1/b[)$  consists of two intervals, given by  $f^{-1}(]0, 1/b[) = (]0, \bar{y}_-[\cup]\bar{y}_+, 1/a[)$  where

$$\bar{y}_{\pm} = f^{-1}(1/b) = \left(\frac{1}{2\sqrt{a}} \pm \frac{1}{2}\sqrt{\frac{1}{a} - \frac{4}{b}}\right)^2 \quad (10)$$

and thus the basin of attraction for the two-dimensional map  $T$  in the phase plane  $(x, y)$  is given by

$$\begin{aligned} \mathcal{B}_T(x_C^*, y_C^*) &= \mathcal{B}_F(x_C^*) \times \mathcal{B}_G(y_C^*) \\ &= ]0, 1/b[ \times \{]0, \bar{y}_-[\cup]\bar{y}_+, 1/a[\} \end{aligned}$$

Fig. 1d shows the basins of attraction in the phase plane. Besides the two rectangles of points (in red) converging to the attracting Cournot point, there are also huge rectangles of points which may be considered undesired points, as leading to the extinction (the origin  $O$ ) or leading to the saddle cycle on the coordinate axes  $C_{2A}$ . Now the separators between the basin of the proper attracting set ( $C$ ) and those in Milnor sense is given by the lines  $x = \frac{1}{b}$  and  $x = \frac{1}{a}$  as before and the lines  $y = \bar{y}_{\pm}$  from the preimages  $f^{-1}(1/b)$ , as clearly visible in Fig. 1d.

This is the main point, in order to have a model well defined in a wider area of the  $(x, y)$  phase plane, we shall modify the basic model, as we shall see in the next Section.

Let us first complete the analysis in the interesting rectangle  $R = [0, 1/b] \times [0, \bar{y}_-]$  (or  $R = [0, \bar{x}_-] \times [0, 1/a]$  in different parameter settings, as we shall explain below) of the region  $Q = [0, 1/b] \times [0, 1/a]$  of the phase plane, which includes the attracting set of the map  $T(x)$ . This part is already known in the literature, however it is suitable to recall it here, in order to remark that the bifurcation of the Cournot point, for the two-dimensional map, is not a flip bifurcation. It is clear that in some way this bifurcation is associated with a flip, as in fact for the map  $F(x)$  the Cournot  $x$ -coordinate undergoes a flip bifurcation. However this is not reflected in a flip bifurcation of the map  $T$ . In our case, a flip-bifurcation of the map  $F(x)$  corresponds to a degenerate Neimark-Sacker bifurcation for  $T$ . In fact, the Jacobian matrix of our map  $T$  in the smooth branches, evaluated in the Cournot fixed point is given by

$$J(x_C^*, y_C^*) = \begin{vmatrix} 0 & \frac{b-a}{2a} \\ \frac{a-b}{2b} & 0 \end{vmatrix} \quad (11)$$

and its characteristic polynomial is given by  $\mathcal{P}(\lambda) = \lambda^2 + D$  where the determinant is  $D = \frac{(a-b)^2}{4ab}$  always positive. Thus the eigenvalues are pure



imaginary. The equilibrium is stable as long as  $|D| < 1$ , which occurs as long as

$$3 - 2\sqrt{2} < r = \frac{b}{a} < 3 + 2\sqrt{2} \quad (12)$$

At the bifurcation value, when  $|D| = 1$ , the eigenvalues are  $\pm i$ . Thus it corresponds to one of the "resonant" cases of the Neimark-Sacker theorem. However, also in such a degenerate case, a closed invariant attracting curve  $\Gamma$  exists after the bifurcation, made up of the saddle-node connection of a pair of 4-cycles. In fact, let us prove this directly via the one-dimensional map  $F(x)$ , for which at the same time a normal flip bifurcation occurs, leading to a locally stable 2-cycle  $\{x_1, x_2\}$ . As we know (see Bischi et al., 2000), a 2-cycle of  $F(x)$ , locally stable, coexisting with the locally unstable Cournot fixed point, leads to the existence of two unstable 4-cycles  $C_{4A}$  and  $C_{4B}$  ( $C_{4A}$  one on the coordinate axes and  $C_{4B}$  with points in the positive quadrant) plus one stable 4-cycle  $C_{4C}$  with points in the positive quadrant. The 2-cycle already existing on the coordinate axes,  $C_{2A}$ , turns into a repelling node. All the periodic points listed above belong to the Cartesian product  $\{0, x_1, x_C^*, x_2\} \times \{0, x_1, x_C^*, x_2\}$ , as we can see in Fig. 2, where the closed curve  $\Gamma$  is also shown, and a portion of the basins of the topological attractors and the attractors in Milnor sense.

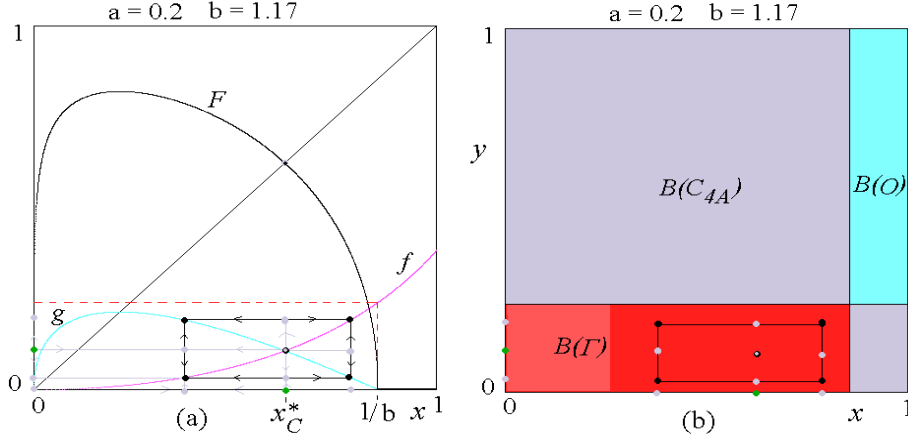


Fig. 2

The transition to chaos for the map  $F(x)$  is as usual, via a sequence of period doubling bifurcations. An example, keeping  $a$  fixed and increasing the parameter  $b$ , is shown in Fig. 3, and the whole sequence can be observed as for the unimodal logistic map up to the last bifurcation, involving the homoclinic bifurcation of the origin.

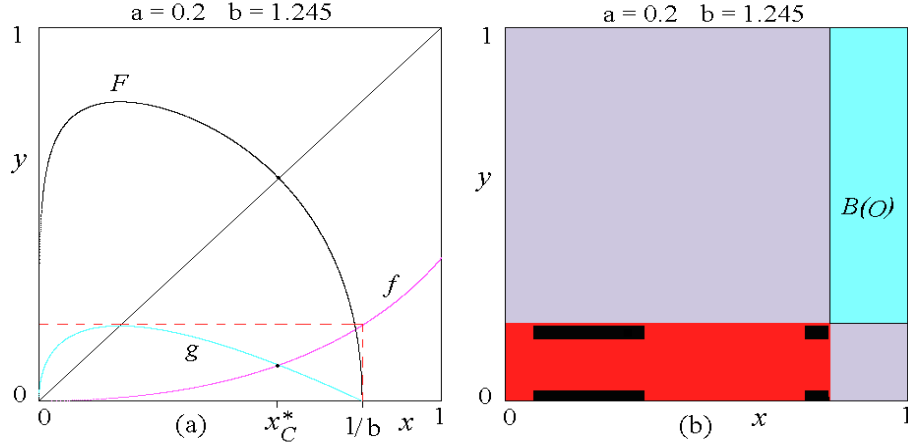


Fig.3

This last bifurcation is illustrated in Fig. 4. As we can see, the map  $F(x)$  has the maximum which ends in the kink point  $1/b$  (and at the same time notice that the maximum of the function  $g(x)$  has a contact with the immediate basin in the line  $y = \bar{y}_-$ ). It follows that for higher values of  $b$  almost all the points of the interval  $]0, 1/b[$  are mapped in the origin. Noticing that the critical point of the smooth function  $g(x)$  is  $x_{cr} = \frac{1}{4b}$  (as  $g'(x_{cr}) = 0$ ) and  $g(x_{cr}) = x_{cr}$ , we have that it is also a critical point of  $F(x)$ , as  $F'(x_{cr}) = 0$  and  $F(x_{cr}) = \sqrt{\frac{1}{4ab}} - \frac{1}{4b}$ . It follows that the final bifurcation of  $F(x)$  (and thus of  $T$ ) occurs when  $F(x_{cr}) = \frac{1}{b}$  which leads to the following condition:

$$\sqrt{\frac{b}{a}} = 2.5 \quad , \quad r = \frac{b}{a} = 6.25 \quad (13)$$

In the case  $a = 0.2$  used in our example we get the final bifurcation at  $b_f = 1.25$ , which is the value used in Fig.4.

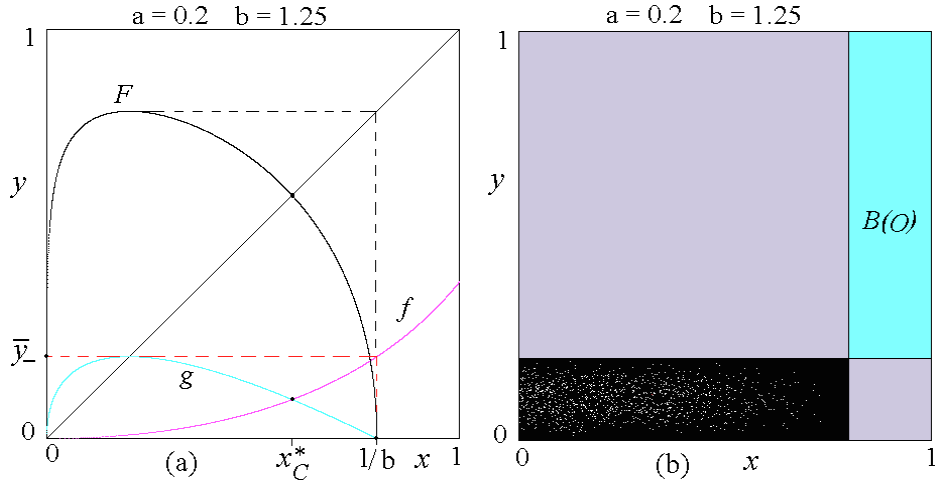


Fig. 4

It is clear that if we change the parameters in such a way that the "adimensional" parameter  $r = \frac{b}{a}$  decreases (instead of increasing it as we have done above) we shall see the roles of the functions exchanged. That is, the first *bifurcation of the basins* occurs when the function  $f(x)$  has a contact with the rectangle  $Q$  and at the same time the bimodal function  $F(x)$  has a contact with zero, and this bifurcation occurs when the parameters satisfy

$$\sqrt{\frac{a}{b}} = 2 \quad , \quad r = \frac{b}{a} = \frac{1}{4} \quad (14)$$

the Neimark-Shaker bifurcations occur at  $r = 3 - 2\sqrt{2}$  and the final bifurcation occurs when the function  $G(y)$  has a contact with  $\frac{1}{b}$  at the following condition:

$$\sqrt{\frac{a}{b}} = 2.5 \quad , \quad r = \frac{b}{a} = \frac{1}{6.25} \quad (15)$$

After the final bifurcation the model is not so quite representative, as only a chaotic repeller survives, and almost all the points of the phase space are mapped into the coordinate axes in a finite number of steps, after which the state jumps from one axis to the other one at each iteration. In the next section we shall analyze a modified model, which is more suitable in the applied context.

Notice that the feasible dynamics observed up to now also correspond to the dynamics of the original smooth model, given by:

$$\begin{aligned} x' = f(y) &= \sqrt{\frac{y}{a}} - y \\ y' = g(x) &= \sqrt{\frac{x}{b}} - x \end{aligned}$$

assuming that the phase space of interest is the rectangle  $Q = [0, 1/b] \times [0, 1/a]$ . That is, all the points outside this range have at least one negative iterate, and thus are considered unfeasible. This model has a *Cournot fixed point which is stable and globally attracting in  $Q$*  as long as the composite functions  $F(x)$  and  $G(y)$  are inside the rectangle  $Q$ , and thus, as we have seen above, only before the first bifurcation of the basins' structure, which holds only in the following range:

$$\frac{1}{4} < r = \frac{a}{b} < 4 \quad (16)$$

Outside this interval, even if we have a locally stable Cournot point, or a different periodic or chaotic attractor, we also have states in the rectangle  $Q$  which lead to some cycle on the coordinate axes. The assumption of a piecewise smooth function as we have assumed in this section, has the effect to allow also states outside the rectangle  $Q$  or inside  $Q$  after the first bifurcation of the basins. However, this result is perhaps not so interesting because the non positive asymptotic states only belong to the coordinate axes. This aspect will be improved in the next section.

It is also worth to mention that there is a symmetry in the model, given by  $T(x, y; a, b) = T(y, x; b, a)$  leading to a symmetric structure of the bifurcation curves in the two dimensional parameter plane  $(a, b)$ , with respect to the line  $a = b$ . This may lead us to reduce of one unit the number of the parameters, and keeping the unique parameter  $r = \frac{b}{a}$ . This requires a rescaling in the variables: setting  $X = bx$  and  $Y = ay$  we obtain a two dimensional map which only depends on  $(X, Y; r)$ ,  $\tilde{T}(X, Y) = (X', Y')$  (which is clearly topologically conjugated with  $T$ ) given by:

$$X' = \begin{cases} r(\sqrt{Y} - Y) & \text{if } 0 \leq Y \leq 1 \\ 0 & \text{if } Y > 1 \end{cases}, \quad (17)$$

$$Y' = \begin{cases} \frac{1}{r}(\sqrt{X} - X) & \text{if } 0 \leq X \leq 1 \\ 0 & \text{if } X > 1 \end{cases}. \quad (18)$$

For a more suitable interpretation of the dynamics in the applied context we prefer to avoid a rescaling in the state variables, so we keep the map in its original form with the parameters  $(a, b)$ , as resulting from the optimization problem. However, as we shall see, a complete analysis performed in the next section is better visualized by using the rescaled map  $\tilde{T}$ .

### 3 The modified Cournot model.

The undesired features of the basic model considered in the previous Section are due to the zero value in the reaction functions, and once that the zero value is get, the iterated states in the duopoly can no longer abandon the coordinate axes, even if the cycles on the axes are all locally unstable. Thus a more interesting model, satisfying the intuitive economic behavior, is that a state variable  $x$  or  $y$  can become very low, assuming a fixed low value, say  $\epsilon$ , after which they can increase again. For sake of simplicity let us take the same constant low value  $\epsilon$  for both competitors. Thus the model we are now considering is the map  $T_\epsilon$ ,  $T_\epsilon(x, y) = (x', y')$  defined as follows:

$$x' = f(y) = \begin{cases} \sqrt{\frac{y}{a}} - y & \text{if } y \leq \frac{1}{a} \\ \epsilon & \text{if } y > \frac{1}{a} \end{cases}, \quad (19)$$

$$y' = g(x) = \begin{cases} \sqrt{\frac{x}{b}} - x & \text{if } x \leq \frac{1}{b} \\ \epsilon & \text{if } x > \frac{1}{b} \end{cases}. \quad (20)$$

which is always piecewise smooth, but now discontinuous, with discontinuity lines in  $x = \frac{1}{b}$  and  $y = \frac{1}{a}$ . It is clear that the analytical results of the previous map inside the rectangle  $Q$  of the phase space are unchanged, and work also for the modified model. However, the main fact is that the zero state can no longer be reached. The one-dimensional function  $F(x) = f(g(x))$  (with a point of discontinuity in  $x = \frac{1}{b}$ ) has now the origin which is really a repelling fixed point, while for  $x > \frac{1}{b}$  the function takes the constant value  $F(x) = f(\epsilon)$  say  $x_m = f(\epsilon) = \sqrt{\frac{\epsilon}{a}} - \epsilon$  which is the minimum value that can be reached by the iterated points of the map, inside the existing absorbing interval. As before, as the parameters  $(a, b)$  are changed increasing  $r$  the final bifurcation occurs when the maximum value of  $F$  reaches  $\frac{1}{b}$ , that is, when (13) holds. And it is also immediate to realize that now the final bifurcation will not lead the dynamics to the axes. Instead, all the states exceeding  $\frac{1}{b}$  are mapped into  $x_m$  which will be a periodic point.

Stated in other words, once that the state variable  $x$  reaches a low value, the increasing branch of  $F(x)$  issuing from the origin will push the state to increase again, entering the absorbing interval with minimum value  $x_m$  and maximum in the critical value of  $F(x)$ . It follows that the trajectories are converging to a cycle different from the fixed point  $O$ , with positive state variables and superstable. Clearly the period of the cycle depends on the value of  $\epsilon$  and on the values of the other parameters.

As an example, let us consider the case at  $a = 0.2$  fixed considered in the previous section. Now, with the new model  $T_\epsilon$ , and assuming  $\epsilon = 10^{-4}$ , the cycles existing for  $b < 1.25$  are exactly the same with the same coordinates up to the final bifurcation, but the basins of attraction of the attracting cycles are now changed. On the coordinate axes there are now truly repellers, which are no longer attractors in Milnor sense (their stable set is a set of zero measure). All the points which we observed before in the basin of the origin or in the basin of some cycle on the coordinate axes, are now converging to the attractors in the positive quadrant of the phase plane.

And after the final bifurcation, for  $b > 1.25$ , when the dynamics were previously no longer interesting, we have now that almost all the trajectories are converging to a superstable cycle, whose period depends on the parameters' values. An example is shown in Fig.5.

Fig. 5a illustrates a two-dimensional bifurcation diagram in the  $(a, b)$  plane. Different colors correspond to cycles of different period of  $F(x)$ . A vertical section is shown through a one-dimensional bifurcation diagram in Fig. 5b, giving the state variable  $x$  as a function of the parameter  $b$ . At

$b = b^*$  the Cournot point becomes unstable and at  $b = b_f$  the maximum of  $F(x)$  reaches the value  $1/b$ . Then we can see that the period of the cycle changes up to a 2-cycle which persists for a wide interval. A change in the period of the trajectories after the final bifurcation in  $Q$  is due to a *border collision bifurcation* with the discontinuity point.

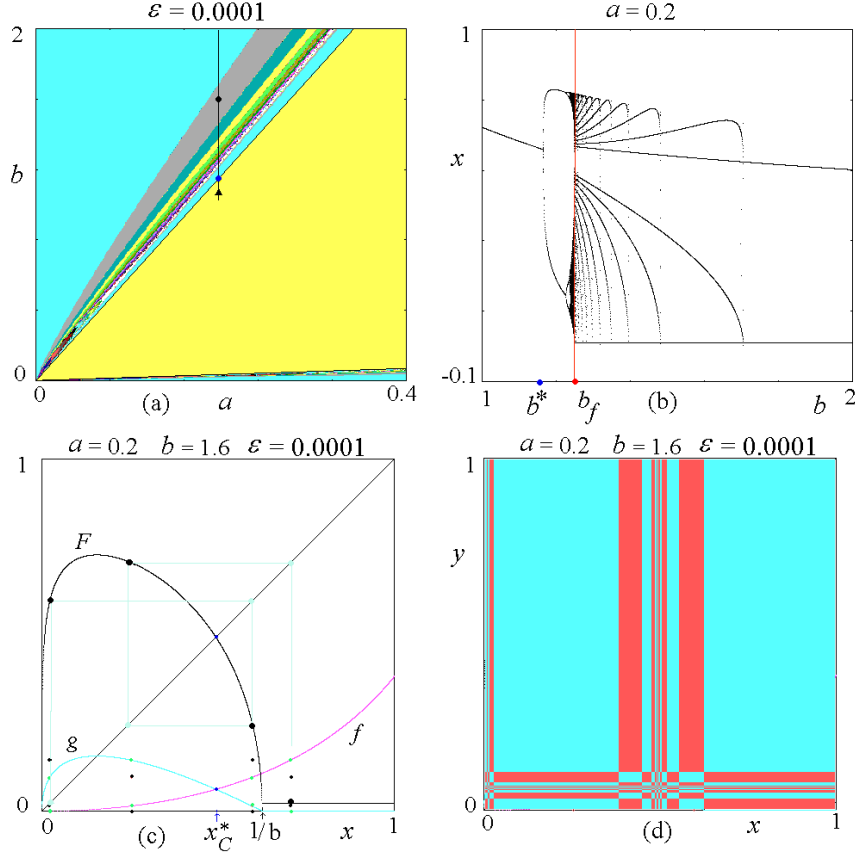


Fig. 5

We remark that the one-dimensional bifurcation diagram shows the  $x$ -variable as a function of the parameter  $b$ , and thus the period there observable is the period for the one-dimensional map  $F(x)$ . This does not correspond to the period of the cycles of the two-dimensional map because there may be more periodic points with the same projection in the coordinate axes (indeed this is the characteristic property of maps like the present one, for which the second iterate has separate variables). An example of the map at  $b = 1.6$  ( $> b_f$ ) for which the function  $F(x)$  has a 4-cycle, is shown in Fig.5c. For the map  $T_\epsilon$  this 4-cycle leads to two disjoint attracting 8-cycles ( $C_{8B}$  and  $C_{8C}$ ). The whole positive phase plane consists of points converging to one or the other of the 8-cycles. The two basins of attraction are shown in Fig.5d. We remark that the rectangles of the basins which are visible in Fig.5b are not finite

in number, as they are accumulating on the coordinate axes and can be seen only in enlarged windows (the structure of the basins in piecewise smooth duopoly games has been described also in Tramontana et al., 2009).

As already remarked above, the period also depends on the choice of the parameter  $\epsilon$ . For example, by using  $\epsilon = 10^{-5}$  we get a different picture, shown in Fig. 6.

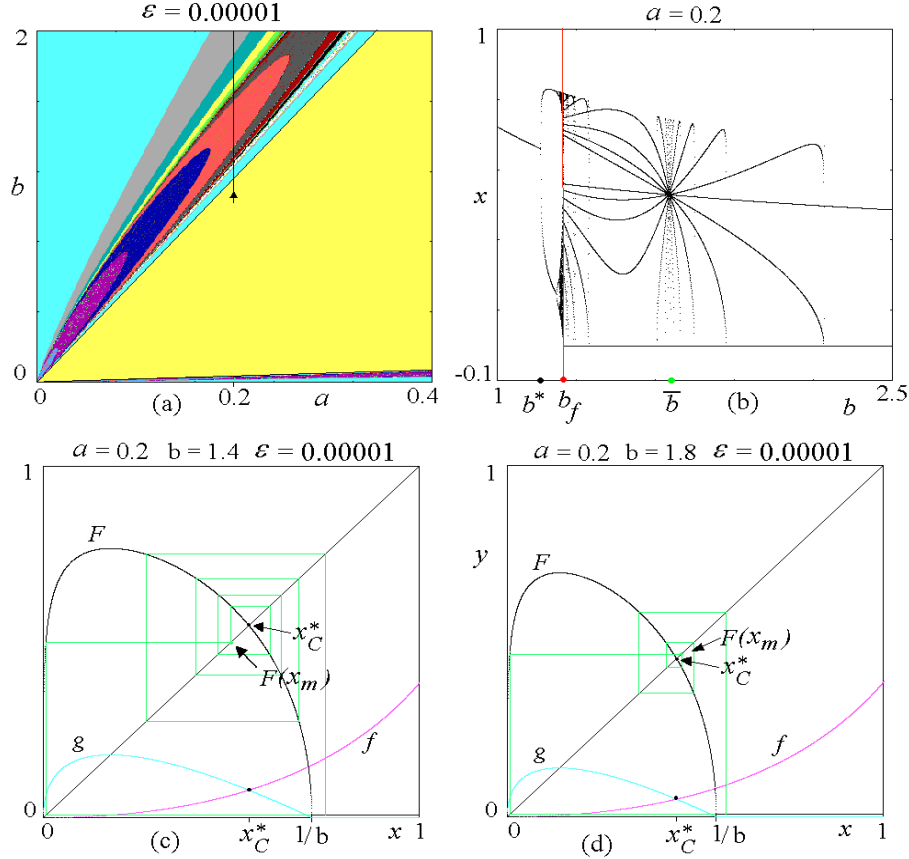


Fig. 6

Fig. 6a illustrates the two-dimensional bifurcation diagram in the  $(a, b)$  parameter plane, while Fig. 6b illustrates a section at  $a = 0.2$  as before. We can see that the periods are changed and also that infinitely many periods can be obtained. In fact, in Fig. 6b we can see that there is a particular bifurcation in the parameter  $\bar{b}$ : the periods are odd and increase by two units at some BCBs as  $b$  tends to  $\bar{b}$  while after  $\bar{b}$  the period is even and decreases by two units. This sequence of bifurcations can be easily explained from the graph of the function  $F(x)$ . In Fig. 6c we can see that for  $b < \bar{b}$  the value  $x_m = f(\epsilon)$  is above the preimage of the Cournot fixed point or, equivalently, its image is below the unstable Cournot fixed point:  $F(x_m) < x_C^*$ . In the example of Fig. 6d, at  $b =$

$1.4 < \bar{b}$  the minimum point is periodic of period 9. As  $b$  increases the point  $F(x_m)$  tends to the fixed point and the period increases because this periodic points must do more and more turns around the unstable Cournot point before reaching again the minimum value. Clearly the period tends to infinity and at the value  $b = \bar{b}$  we have  $F(x_m) = x_C^*$  that is: the minimum value is preperiodic to the Cournot point. Then for  $b > \bar{b}$  we have  $F(x_m) > x_C^*$  and  $F(x_m)$  increases, so that the period from very high tends to decrease (in the example shown in Fig.6d at  $b = 1.8$  the minimum point is periodic of period 6).

In both the examples shown in Fig. 5b and Fig. 6b, at  $b = b^*$  (given in (12)) the local bifurcation of the Cournot fixed point occurs while at  $b = b_f$  (given in (13)) the final bifurcation occurs. The main property of the map after the final bifurcation is that almost all converges to the existing cycle for the map  $F(x)$  which is *unique and superstable*. In fact, almost all the points inside the interval  $]0, 1/b[$  exit from that interval in a finite number of steps under  $F(x)$  and ultimately take the value  $x_m$  which then is periodic of some period. It follows that almost all the trajectories converge to this cycle, which is *superstable*, having one periodic point in a flat branch of  $F(x)$  with zero derivative.

An attracting cycle of period  $p$  for  $F(x)$  corresponds to several co-existing attracting cycles for the two-dimensional map  $T_\epsilon$ , following the rules explained in Bischi et al. (2000).

Clearly the whole analysis with obvious changes occurs if the parameters are changed such that  $r$  moves in the opposite direction, decreasing.

### 3.1 Dependence on $\epsilon$ .

The examples shown above illustrate that the positive constant value in the graph of the reaction functions is important in order to have a representative model. Also evidence that the dependence of the periods from the value of  $\epsilon$  is very strong. To better investigate this dependence let us consider the topologically conjugated model as a function of the only parameter  $r = b/a$ , so that, considering  $\epsilon$  as a second parameter, we may plot a two-dimensional bifurcation diagram in the parameter plane  $(r, \epsilon)$ .

By using the change of coordinates  $X = bx$  and  $Y = ay$  we obtain the map  $\tilde{T}_\epsilon$  which only depends on  $(X, Y; r, \epsilon)$ ,  $\tilde{T}_\epsilon(X, Y) = (X', Y')$  given by:

$$X' = \begin{cases} r(\sqrt{Y} - Y) & \text{if } 0 \leq Y \leq 1 \\ \epsilon & \text{if } Y > 1 \end{cases}, \quad (21)$$

$$Y' = \begin{cases} \frac{1}{r}(\sqrt{X} - X) & \text{if } 0 \leq X \leq 1 \\ \epsilon & \text{if } X > 1 \end{cases}. \quad (22)$$



Fig.7 shows the dependence on the constant value  $\epsilon$ , for  $\epsilon \in [0, 0.06]$  in Fig. 7a, while in the enlarged window  $\epsilon \in [0, 0.0014]$ .

At  $r = r^*$  (given in (12)) the local bifurcation of the Cournot fixed point occurs. At  $r = r_f$  (given in (13)) the final bifurcation in  $Q$  occurs. In the interval  $r^* < r < r_f$  the bifurcations are those of a one-dimensional unimodal map, independent on  $\epsilon$ , and the bifurcations are vertical lines. The new interesting range is for  $r > r_f$ . We can see that all the periods can be detected, in fact, the periodicity regions in the enlargement follow the structure of the box-within-a box bifurcation of the unimodal maps (see Mira, 1987), but now applied only to superstable cycles which change their period via border collision bifurcations. We remark that the periodicity regions cannot overlap because at fixed parameters, as we have seen, it is possible to have only one superstable cycle.

It is plain that a graph similar to the one in Fig. 7 can be obtained also decreasing  $r$ .

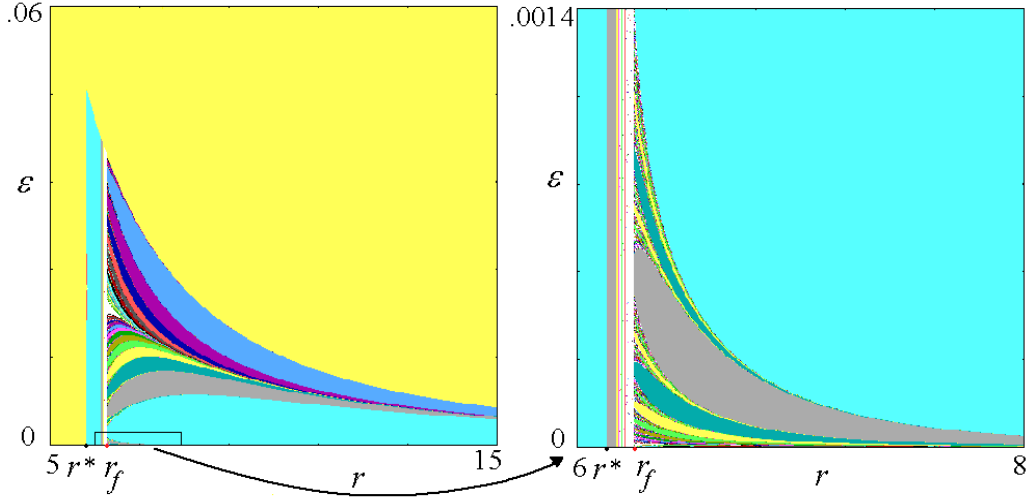


Fig. 7

## 4 Conclusions

The well known Cournot duopoly model has been here investigated with respect to its global properties. Also when the Cournot fixed point is locally stable or another attracting set exists which lead to interesting dynamics, there may be states in the phase space which lead to dangerous situations (negative or zero production). The assumption of a positive minimal quantity in the reaction functions has the effect to enlarge the region in the phase space associated with feasible dynamics, mainly periodic. Specially in extreme cases, after the final bifurcation for the bounded dynamics in a neighborhood of the Cournot fixed point,

we have proved the existence of superstable cycles of any period, with positive quantities periodically changed, which attract almost all the points in the phase space (i.e. except for a set of zero measure, which may include a repelling Cantor set of points)

We remark that the results evidenced in the last two sections are not due to the introduced discontinuity in the shape of the reaction function. They only depend on the positive horizontal graph of the reaction functions. In fact, the results and comments of the last two sections are still valid in a continuous piecewise smooth model, in which the kink points are not assumed at  $x = 1/b$  and  $y = 1/a$  but are assumed dependent on the choice of  $\epsilon$  still keeping continuous the model, as follows:

$$x' = f(y) = \begin{cases} \sqrt{\frac{y}{a}} - y & \text{if } y \leq p_y \\ \epsilon & \text{if } y > p_y \end{cases}, \quad (23)$$

$$y' = g(x) = \begin{cases} \sqrt{\frac{x}{b}} - x & \text{if } x \leq p_x \\ \epsilon & \text{if } x > p_x \end{cases}. \quad (24)$$

where the kink points  $p_x$  and  $p_y$  satisfy the conditions leading to continuous reaction functions:

$$\epsilon = \sqrt{\frac{p_y}{a}} - p_y, \quad \epsilon = \sqrt{\frac{p_x}{b}} - p_x$$

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